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Diffusive thermal conductivity of superfluid ^{3}He-A at low temperatures

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Abstract. The components of the diffusive thermal conductivity tensor of superfluid ^{3}He-A are calculated by using approximate collision integrals at low temperatures. The energy and temperature dependence of the quasi-particle relaxation time are obtained. The parameter $\bar{\lambda}_1$ plays an important role in determining the temperature dependence of the diffusive thermal conductivity coefficients.

1. Introduction

Since the discovery of superfluid phases of ^{3}He, investigation of the coefficients of the diffusive thermal conductivity of the A phase has received less attention. Most theoretical efforts have been concentrated on the evaluation of the diffusive thermal conductivity of the B phase. In [1, 2] the Boltzmann equation was solvent in the low-temperature limit for thermal conductivity for this phase exactly. It was found that the diffusive thermal conductivity varies with temperature as T^{-1}, the same as in the normal state. In [3], by using an approximate collision integral which gives nearly exact results in the limits $T \to 0$ and $T \to T_c$, this coefficient was obtained for the whole range of temperatures numerically. In [4], by choosing an appropriate trial solution, the Boltzmann equation was solved variationally for the diffusive thermal conductivity of the B phase.

The purpose of this paper is to use the approximate collision integral in [3] and to modify it for the diffusive thermal conductivity of the A phase of liquid ^{3}He. The components of the diffusive thermal conductivity tensor are formulated in terms of the Bogoliubov quasi-particle relaxation time for the whole range of temperatures. The evaluation of these components against temperature needs numerical calculations which we defer to elsewhere and here we compute them at low temperatures. In § 2, we formulate the problem, the components of diffusive thermal conductivity are obtained at low temperatures in § 3. Section 4 is allocated to discussion and concluding remarks.

2. Formulation of the problem

The diffusive thermal conductivity tensor K_{ij} is defined by

$$J_i = K_{ij}\, \partial T/\partial \boldsymbol{r}_j \qquad (1)$$

where $\boldsymbol{J}$ is the diffusive heat current. It is noted that, in addition to heat transfer by a

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random diffusive process of the thermal excitations described by equation (1), there is a convective contribution to the heat current in a superfluid, even in the absence of mass flow, owing to the possibility of normal–superfluid counterflow. At low temperatures, this contribution is negligible [3, 5] and the measurement of the components of thermal conductivity at these temperatures is more meaningful than the measurement of the thermal conductivity components in the vicinity of T_c.

The diffusive heat current may be written in terms of the quasi-particle distribution functions δn_p as

$$J = \sum_p E_p V_p \, \delta n_p' \tag{2}$$

where V_p is the Bogoliubov quasi-particle velocity and $\delta n_p' = \delta n_p - n_p' \, \delta E_p$ characterises the deviation from local equilibrium. In the presence of the stationary diffusive heat current the streaming term in the Boltzmann equation may be written as

$$[(\boldsymbol{p} \cdot \boldsymbol{q})/m^*](\varepsilon_p/E_p) n_p' (E_p/T) \, \delta T. \tag{3}$$

Following [3], the collision integral may be expressed as

$$-(\mathrm{i}/\tau_p) \, \delta n_p' + I_p^{\mathrm{in}} \tag{4}$$

where τ_p is the quasi-particle relaxation time, and the 'in scattering' term I_p^{in} of the collision integral, is written for the A phase as

$$I_p = \frac{\bar{\lambda}_1}{\tau_p} \sum_{m=-1}^{1} \left(n_p' \varepsilon_p Y_{1m}(\hat{\boldsymbol{P}}) \sum_k Y_{1m}^*(\hat{\boldsymbol{k}}) \varepsilon_k \frac{\delta n_k'}{\tau_k} \Big/ \sum_k |Y_{1m}(\hat{\boldsymbol{k}})|^2 \frac{n_k'}{\tau_k} E_k^2 \right) \tag{5}$$

where

$$\bar{\lambda}_1 = 1 + (2/\langle w \rangle) \langle w(\theta, \varphi) \cos\theta \rangle \tag{6}$$

with

$$\langle A \rangle = \int \frac{\mathrm{d}\Omega}{4\pi} \frac{A(\theta, \varphi)}{\cos(\theta/2)}.$$

Introducing a dimensionless function φ_p by

$$\delta n_p' = \mathrm{i}[(\boldsymbol{p} \cdot \boldsymbol{q})/m^*](\varepsilon_p/E_p) n_p' (E_p/Tk_{\mathrm{B}}) \delta T \, \varphi_p \tag{7}$$

the Boltzmann equation may be written as

$$\tau_p = \varphi_p - \frac{\bar{\lambda}_1}{3} \sum_{m=-1}^{1} \left(Y_{1m}(\hat{\boldsymbol{P}}) \sum_k Y_{1m}^*(\hat{\boldsymbol{k}}) \varepsilon_k^2 \boldsymbol{k} \cdot \boldsymbol{q} \frac{n_k'}{\tau_k} \varphi_k \Big/ \frac{\boldsymbol{p} \cdot \boldsymbol{q}}{m^*} \sum_k |Y_{1m}(\hat{\boldsymbol{k}})|^2 E_k^2 \frac{n_k'}{\tau_k} \right). \tag{8}$$

By substituting equation (7) in equation (2) and comparing with equation (1), we have

$$K_{ij} = -(3n/Tm^*) \langle\!\langle \hat{P}_i \hat{P}_j \varepsilon_p^2 n_p' \varphi_p \rangle\!\rangle \tag{9}$$

where

$$\langle\!\langle A \rangle\!\rangle \equiv \int \frac{\mathrm{d}\Omega_p}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\varepsilon_p \, A(\theta, \varphi).$$

From equations (8) and (9), after some algebra, we obtain finally

$$K_{ij} = -\frac{3n}{m^* T} \Bigg\{ \langle\!\langle \hat{P}_i \hat{P}_j \varepsilon_p^2 n_p' \tau_p \rangle\!\rangle + \frac{\bar{\lambda}_1}{3} \sum_{m=-1}^{1} \frac{\langle\!\langle \hat{P}_i Y_{1m}(\hat{\boldsymbol{P}}) \varepsilon_p^2 n_p' \rangle\!\rangle \langle\!\langle \hat{P}_j Y_{1m}^*(\hat{\boldsymbol{P}}) \varepsilon_p^2 n_p' \rangle\!\rangle}{\langle\!\langle |Y_{1m}(\hat{\boldsymbol{P}})|^2 E_p^2 \, n_p'/\tau_p \rangle\!\rangle}$$
$$\times \left[1 - \frac{\bar{\lambda}_1}{3} \left(\left\langle\!\!\left\langle |Y_{1m}(\hat{\boldsymbol{P}})|^2 \varepsilon_p^2 \frac{n_p'}{\tau_p} \right\rangle\!\!\right\rangle \Big/ \left\langle\!\!\left\langle |Y_{1m}(\hat{\boldsymbol{P}})|^2 E_p^2 \frac{n_p'}{\tau_p} \right\rangle\!\!\right\rangle \right) \right]^{-1} \Bigg\}. \tag{10}$$

As we said previously, for evaluation of the diffusive thermal conductivity coefficient against temperature, one needs to use numerical methods which we defer to elsewhere. In § 3, we evaluate them at low temperatures.

3. The components of the diffusive thermal conductivity tensor at low temperatures

The diffusive thermal conductivity tensor for a system with uniaxial symmetry can be written in terms of the components of the symmetry axis $\hat{l}$ with two coefficients $K_\parallel$ and $K_\perp$:

$$K_{ij} = K_\parallel \hat{l}_i \hat{l}_j + K_\perp(\delta_{ij} - \hat{l}_i \hat{l}_j). \tag{11}$$

By taking the polar axis along $\hat{l}$, we have $K_\parallel = K_{zz}$ and $K_\perp = K_{xx} = K_{yy}$. To compute these coefficients at low temperatures, say $T_c/T_F \ll T/T_c \ll 1$, we use the fact that the function n'_p in the integrands of equation (10) is almost non-zero only for the values of $\sin\theta_p \simeq 0$ since $\varepsilon_p \sim k_B T$. In the following, we first calculate the Bogoliubov quasi-particle relaxation time τ_p for the ABM state at low temperatures.

In a normal Fermi liquid the total quasi-particle number is conserved and therefore the only allowed scattering processes are those in which the number of quasi-particles in the final state is the same as the number in the initial state. At low temperatures the density of excitations is low, and consequently the most important processes are those in which two quasi-particles scatter. The quasi-particle relaxation time can therefore be written as [6]

$$\tau_{p_1}^{-1} = \sum_{2,3,4} W(p_1, \ldots, p_4)\delta(\boldsymbol{P}_1 + \boldsymbol{P}_2 - \boldsymbol{P}_3 - \boldsymbol{P}_4)\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\delta_{\sigma_1+\sigma_2,\sigma_3+\sigma_4} \times n_2^0(1 - n_3^0)(1 - n_4^0). \tag{12}$$

By following the analysis of [7], equation (12) transforms into

$$\tau_{p_1}^{-1} = \frac{(m^*)^3}{(2\pi)^6}\int \frac{W(\theta,\varphi)\sin\theta}{\cos(\theta/2)}\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\varphi_2\,\mathrm{d}\varepsilon_2\,\mathrm{d}\varepsilon_3\,\mathrm{d}\varepsilon_4\,\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)n_2^0(1 - n_3^0) \times (1 - n_4^0) \tag{13}$$

where the integration over the energy variables can be done exactly [8], θ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and φ is the angle between the planes spanned by $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ and $(\boldsymbol{p}_3, \boldsymbol{p}_4)$.

In a superfluid the quasi-particle number is not conserved and other processes as well as two-quasi-particle scattering processes can occur. For example, one quasi-particle can decay into three, or three quasi-particles can coalesce to produce one. The interaction between the quasi-particles in the superfluid can be found by performing a Bogoliubov transformation on the normal-state interaction. The Bogoliubov transformation between the normal quasi-particle creation operator $a^+_{p,\sigma}$ and annihilation operator $a_{p,\sigma}$ and the creation operator $\alpha^+_{p,\sigma}$ and annihilation operator $\alpha_{p,\sigma}$ in the superfluid state is

$$a_{p,\sigma} = \sum_\beta U^p_{\sigma\beta}\alpha_{p,\beta} - V^p_{\sigma\beta}\alpha^+_{-p,\beta} \qquad a^+_{p,\sigma} = \sum_\beta U^{p*}_{\sigma\beta}\alpha^+_{p,\beta} - V^{p*}_{\sigma\beta}\alpha_{-p,\beta} \tag{14}$$

where the matrix elements $U^p_{\sigma\beta}$ and $V^p_{s\beta}$ can be chosen for the ABM state as

$$U^p_{\alpha\beta} = [\tfrac{1}{2}(1 + \varepsilon_p/E_p)]^{1/2}\,\delta_{\alpha\beta} \qquad V^p_{\alpha\beta} = [\tfrac{1}{2}(1 - \varepsilon_p/E_p)]^{1/2}\,\delta_{\alpha\beta} \tag{15}$$

where $E_p = (\varepsilon_p^2 + |\Delta_p|^2)^{1/2}$, $|\Delta_p| = \Delta(T)\sin\theta_p$, $\Delta(T)$ is the maximum gap and θ_p is the angle between the quasi-particle momentum and gap axis $\hat{l}$. Hence, if the Hamiltonian for the binary collision process in the normal state is written in terms of the Bogoliubov quasi-particle operators $\alpha_{p,\sigma}$, one can easily see that

$$H = \tfrac{1}{4} \sum_{1,2,3,4} \langle 3,4|t|1,2\rangle (U_4^* \alpha_4^+ - V_4^* \alpha_{-4})(U_3^* \alpha_3^+ - V_3^* \alpha_{-3}) \times (u_1 \alpha_1 - V_1 \alpha_{-1}^+)(U_2 \alpha_2 - V_2 \alpha_{-2}^+) \tag{16}$$

which contains terms $\alpha_4^+ \alpha_2^+ \alpha_{-2}^+ \alpha_1$, $\alpha_4^+ \alpha_1 \alpha_{-3} \alpha_2$, $\alpha_4^+ \alpha_3^+ \alpha_2 \alpha_1$, $\alpha_4^+ \alpha_3^+ \alpha_{-2}^+ \alpha_{-1}^+$ and $\alpha_{-4}\alpha_{-3}\alpha_2\alpha_1$. These terms convert a quasi-particle into three, convert three quasi-particles into one, convert two quasi-particles into two, create four quasi-particles from the condensate and scatter four quasi-particles into the condensate, respectively. The last two processes are not allowed, because in each process the total energy should be conserved. The linearised collision term in the Boltzmann equation due to the two-quasi-particle scattering, decay and coalesce processes, respectively, may be written as

$$I_{22} = \frac{(m^*)^3}{(2\pi)^6} \int \frac{W_{22} \sin\theta}{\cos(\theta/2)} \mathrm{d}\theta\, \mathrm{d}p\, \mathrm{d}\varphi_2\, \mathrm{d}\varepsilon_2\, \mathrm{d}\varepsilon_3\, \mathrm{d}\varepsilon_4\, \delta(E_1 + E_2 - E_3 - E_4) \times n_2^0(1 - n_3^0)(1 - n_4^0) \tag{17}$$

$$I_{13} = \frac{(m^*)^3}{(2\pi)^6} \int \frac{W_{13} \sin\theta}{\cos(\theta/2)} \mathrm{d}\theta\, \mathrm{d}p\, \mathrm{d}\varphi_2\, \mathrm{d}\varepsilon_2\, \mathrm{d}_3\, \mathrm{d}\varepsilon_4\, \delta(E_1 - E_2 - E_3 - E_4) \times (1 - n_2^0)(1 - n_3^0)(1 - n_4^0) \tag{18}$$

$$I_{31} = \frac{(m^*)^3}{(2\pi)^6} \int \frac{W_{31} \sin\theta}{\cos(\theta/2)} \mathrm{d}\theta\, \mathrm{d}\varphi\, \mathrm{d}\varphi_2\, \mathrm{d}\varepsilon_2\, \mathrm{d}\varepsilon_3\, \mathrm{d}\varepsilon_4\, \delta(E_1 + E_2 + E_3 - E_4) \times n_2^0 n_3^0(1 - n_4^0) \tag{19}$$

where the transition probability of the decay process, for example, may be written as

$$W_{13} = |\langle \ldots; P_3, \sigma; P_4, \sigma'; -P_2, -\sigma'; \ldots |V| \ldots; P_1, \sigma; \ldots \rangle|^2 \tag{20}$$

where, in general, V can be written as

$$V = \sum_{1,2,3,4} V_q (U_4^* \alpha_4^+ + V_4^* \alpha_{-4})(U_3^* \alpha_3^+ + V_3^* \alpha_{-3})(U_1 \alpha_1 + V_1 \alpha_{-1}^+) \times (U_2 \alpha_2 + V_2 \alpha_{-2}^+). \tag{21}$$

By using equations (21) and (15) in equation (20) and in the similar expressions for W_{22} and W_{31}, after straightforward lengthy calculations we get

$$\begin{aligned}
W_{22}(\uparrow\downarrow) &= (|V_0|^2/4)[1 - (|\Delta_3||\Delta_2| - \varepsilon_3\varepsilon_2)/E_3E_2][1 - (|\Delta_1||\Delta_4| - \varepsilon_1\varepsilon_4)/E_1E_4] \\
W_{22}(\uparrow\uparrow) &= (|V_0|^2/4)[1 - (|\Delta_4||\Delta_3| + \varepsilon_3\varepsilon_4)/E_4E_3][1 - (|\Delta_2||\Delta_1| + \varepsilon_2\varepsilon_1)/E_2E_1] \\
W_{13}(\uparrow\downarrow) &= (|V_0|^2/4)[1 - (|\Delta_3||\Delta_2| + \varepsilon_3\varepsilon_2)/E_3E_2][1 - (|\Delta_4||\Delta_1| - \varepsilon_4\varepsilon_1)/E_4E_1] \\
W_{13}(\uparrow\uparrow) &= (|V_0|^2/4)[1 - (|\Delta_4||\Delta_3| + \varepsilon_4\varepsilon_3)/E_4E_3][1 - (|\Delta_2||\Delta_1| - \varepsilon_2\varepsilon_1)/E_2E_1] \\
W_{31}(\uparrow\downarrow) &= (|V_0|^2/4)[1 - (|\Delta_2||\Delta_1| + \varepsilon_2\varepsilon_1)/E_2E_1][1 - (|\Delta_4||\Delta_3| - \varepsilon_4\varepsilon_3)/E_4E_3] \\
W_{31}(\uparrow\uparrow) &= (|V_0|^2/4)[1 - (|\Delta_4||\Delta_1| - \varepsilon_4\varepsilon_1)/E_4E_1][1 - (|\Delta_3||\Delta_2| + \varepsilon_3\varepsilon_2)/E_3E_2]
\end{aligned} \tag{22}$$

where we put $\boldsymbol{q} = \mathbf{0}$. It can be shown [9] that, at low temperatures, $\sin\theta_{pi} \simeq 0$ ($i = 1, 2, 3, 4$). By substituting $\sin\theta_{pi} \simeq 0$ into equation (22), we see that only $W_{22}(\uparrow\downarrow) = |V_0|^2$ is non-zero and the other transition probabilities are zero; hence the two-quasi-particle process dominates in the collision integrals at low temperatures. The expression for $\tau_{p_1}^{-1}|\theta_{p_1} \simeq 0$ at low temperatures is therefore nearly the same as the normal state with the exception that the integration on the angle θ is now from zero to θ_m (for the value of θ_m see below). The integration on energies can be done exactly [8] and we have

$$\tau_{p_1}^{-1}|\theta_{p_1} \simeq 0 = (\pi\theta_m^2/256\varepsilon_F)A_s^2[(\pi k_B T)^2 + E_{p_1}^2]/[1 + \exp(-E_{p_1}/k_B T)] \tag{23}$$

where the dimensionless singlet component $A_s = \Sigma_l S_l$ can be expressed in terms of the spin symmetric (and anti-symmetric) Landau parameters F_l^s(and F_l^a) by $S_l = A_l^s - 3A_l^a$ with

$$A_l^{s,a} = F_l^{s,a}/[1 + F_l^{s,a}/(2l + 1)]. \tag{24}$$

By comparison of the terms E_{p_1} and $\pi k_B T$ in the numerator of equation (23), we may write [10] $E_{p_1} \simeq \pi k_B T$, and $\Delta(0)\theta_{p_{1m}} = \pi k_B T$, which gives $\theta_m \simeq \theta_{p_{1m}} \simeq \pi k_B T/\Delta(0)$.

In [11] and in [12] on the basis of a scaling argument, and using equations (12) and (13), respectively, $\tau_{p_1}^{-1}$ was estimated to be proportional T^4, which can be obtained from equation (23) if we put $E_{p_1} = 0$ and $\theta_m \propto T$. At extremely low temperatures, say $T/T_c \leqslant T_c/T_F$, in [11] on the basis of a scaling argument and using equation (12), $\tau_{p_1}^{-1}$ was found to be proportional to T^5, but our approach in obtaining τ_{p_1} cannot be used in this range of temperatures.

By substituting τ_{p_1} from equation (23) into equation (10) and carrying out the integration over $t = \beta E_p$ numerically, we have

$$K_\parallel = 6.17(nV_F^2/A_s^2 T)[1 + 0.72\,\lambda_1^-/(3 - \lambda_1^-)] \tag{25}$$

$$K_\perp = 15.24(nV_F^2/A_s^2 T)[(k_B T)^2/\Delta^2(0)][1 + 0.72\,\lambda_1^-/(3 - \lambda_1^-)]. \tag{26}$$

In [12] by using a simple relaxation time, i.e. $\lambda_1^- = 0$, and on the basis of a scaling argument for the evaluation of the temperature dependence of τ_p, it was estimated that $K_\parallel \propto T^{-2}$ and $K_\perp \propto T^0$. Presumably the difference between these results and equations (25) and (26) with $\lambda_1^- = 0$ might come from the fact that in [12] a different formula was used for K_{ij} from ours (equation (10) (with $\lambda_1^- = 0$)); this is not clear from their letter [12].

4. Discussion and concluding remarks

As we discussed previously, the measurement of the components of thermal conductivity is more meaningful at low temperatures than in the vicinity of T_c. The thermal coefficients $K_\parallel$ and $K_\perp$ have been obtained in equations (25) and (26) with temperature dependences T^{-1} and T, respectively, at low temperatures if λ_1^- is taken to be a constant parameter with respect to temperatures (see below). These results can be estimated on the bases of a scaling argument from equations (13) and (10). As is clear from equations (25) and (26), the diffusive thermal conductivity components depend on the values of the parameter λ_1^- (defined in equation (6)). In the calculations of the diffusive thermal conductivity of the B phase in [3], λ_1^- (which can be determined from other experiments) was used as a parameter. In [2] the various theoretical results for the normalised quantity $KT/K_c T_c$ were collected as a function of reduced temperature; these are strongly dependent on the value of λ_1^-. The various values of λ_1^- arise because of the different sources of evaluation of the Landau parameters. Despite the temperature independence of λ_1^- in the B phase, in the A phase the values of λ_1^- at low temperatures ($T/T_c \ll 1$) play an important role in determining the temperature dependence of the thermal coefficients $K_\parallel$ and $K_\perp$. By using equation (6) and keeping in mind that the variable θ varies between zero and $\theta_m \simeq \pi k_B T/\Delta(0)$, we get

$$\lambda_+^- = 3 - (\pi k_B T)^2/2\Delta^2(0). \tag{27}$$

By substituting equation (27) in equations (25) and (26), we have, for low temperatures ($T_c/T_F \ll T/T_c \ll 1$),

$$K_\parallel = 2.70\, nV_F^2 \Delta^2(0)/A_s^2 k_B^2 T^3 \tag{28}$$

$$K_\perp = 6.67\, nV_F^2/A_s^2 T. \tag{29}$$

It is interesting to note that the temperature dependence of $K_\perp$ at low temperatures is the same as the diffusive thermal conductivity of the B phase. The temperature dependence of the thermal coefficients $K_\parallel$ and $K_\perp$ at extremely low temperatures ($T/T_c \leqslant T_c/T_F$) can be estimated from equation (10) and equation (12) by using the scaling argument. The results are $K_\parallel \propto T^{-4}$ and $K_\perp \propto T^{-2}$.

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References

[1] Pethick C J, Smith H and Bhattacharyya P 1977 *Phys. Rev.* **15B** 3384
[2] Einzel D 1984 *J. Low Temp. Phys.* **54** 427
[3] Wölfle P and Einzel D 1978 *J. Low Temp. Phys.* **32** 39
[4] Hara J 1981 *J. Low Temp. Phys.* **43** 533
[5] Sykes J and Brooker G A 1970 *Ann. Phys., NY* **56** 1
[6] Pines D and Noziers P 1966 *The Theory of Quantum Liquids* (New York: Benjamin)
[7] Abrikosov A A and Khalatnikov I M 1959 *Rep. Prog. Phys.* **22** 329
[8] Morel P and Noziers P 1962 *Phys. Rev.* **126** 1909
[9] Shahzamanian M A 1989 to be published
[10] Greaves N A and Leggett A J 1983 *J. Phys. C: Solid State Phys.* **16** 4383
[11] Combescot R 1975 *Phys. Rev. Lett.* **35** 471
[12] Valls O and Houghton A 1975 *Phys. Lett.* **54A** 143